On Generalized Relativity in Connection with Mr. W. J. Johnston's Symbolic Calculus.

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The calculus presented by Mr. Johnston in the note printed above is so concise for the direct expression of the relations concerned in electrodynamic relativity, and involves so aptly the main relations of physics, that it may be profitable to consider it in further detail from that point of view. Maxwell had himself introduced quaternion notation into electrodynamics.

In the first place, as to how the essential idea of invariance can come in, it will suffice to illustrate from three dimensions, in which the vector to a point xyz is represented by the binary form $ix+jy+kz$. Hence i, j, k are symbolic units defined algebraically by the typical relations $i^2 = -1$, $ij = -ji$; but the further quaternionic relation $ijk = -1$ is not assumed, for though $(ijk)^2$ is necessarily unity, just as i^2 is -1, the use of roots or fractional powers of operators is excluded. Now just as we can pass from a point xyz to another point $x'y'z'$, so also we can change from one trihedral set ijk to any other one $i'j'k'$ possessing the same characteristic modes of combination. In fact let $x'y'z'$ represent the same point xyz referred to this other set of Cartesian axes, their values being expressed by the usual equations for change of axes; then $ix+jy+kz$ becomes on substitution $i'x'+j'y'+k'z'$, and it is readily verified that the set $i'j'k'$ thus determined is of the same type as ijk. The formulæ so obtained for them are one general specification of such systems of units, irrespective of what value the product ijk may have. Extension of these ideas to n dimensions is natural and direct.

A self-consistent ijk... n-fold algebra has been developed, on geometric analogy, mainly by Clifford* after Hamilton and Grassmann. In it the square of the difference of the vectors of two points, viz. $(i\xi+j\eta+k\zeta+\ldots)^2$ where ξ represents x_1-x_2, etc., turns out to be the purely scalar quantity $-(\xi^2+\eta^2+\zeta^2+\ldots)$, and this suggest the geometrical interpretation. For it is the square, with changed sign, of the *distance* between the two points in Euclidean geometry, and so is invariant when the region is referred to a new set of unitary vectors. Now, the duality inherent in relative properties makes reference to a new trihedron $i'j'k'$ equivalent to a rotation of the region, referred to the previous one ijk, except as regards the possible addition of a reflexion of the region in

* 'Amer. Journ. Math.,' vol. i (1878); or W. K. Clifford's 'Mathematical Papers,' pp. 266–276.

a plane, which need not now be attended to. Thus the square of the distance expressed in this form is invariant for all displacements of the region, as referred to any frame ijk; and this is sufficient to ensure that the geometry inherent in this consistent algebraic foundation is Euclidean, whatever be the number of dimensions. Invariant relations in the algebra connote constructions in that geometry, expressed directly without regard to any system of axes: all such quantities and relations as are evolved in Mr. Johnston's note are of this type, transcending special systems of reference.

An algebra of this kind remains valid when the scalar quantities involved in it, such as $x, y, \ldots$ F, G, ..., are complex quantities instead of real, and this is a main feature of the application to physical fields. The $\sqrt{-1}$ of the complex scalar cannot become mixed or confused with $i, j, k, \ldots$; although their squares are the same -1, their products remain explicit. Thus it is the same as if the scalar symbols were made complex only after the operations are finished.

The equations of electrodynamics are invariant as explained above for changes from any quaternary system $ijko$ to any other system $i'j'k'o'$, when one of the scalar coefficients, that expressing the time, is a pure imaginary. On this scheme a displacement in which time is implicated as the fourth dimension is the same thing as an alteration of translational velocity, in ordinary space, of the system as a whole. Just to the extent that invariance holds, is the formulation thus independent of any *constant translatory* velocity that may be ascribed to the whole system. This statement represents the limited degree of generalization that can arise on the usual scheme and be interpreted as non-essential of space and time, in a calculus like the present one.

In combining results of rotations about the same axis, it is the angles of rotation that are additive in the formulæ: thus, as Mr. A. A. Robb has pointed out, though velocities of convection in the same direction, such as v, are not additive, the corresponding angles represented by $\sinh^{-1}v/c$ become so.

Having thus an auxiliary geometry of n dimensions of Euclidean type, it is the invariants, that arise in its calculus, that are the types of pure self-contained geometrical entity appropriate to such a hyperspace. The object of the Hamiltonian quaternion was to get rid of arbitrary frames of reference; the quantities that alone can occur in a quaternion analysis thus represent these invariants, viz., quantities resulting from additions and multiplications of complete vectors: and the question for that calculus is whether these suffice for geometrical reasonings, or whether an underlying scheme of reference such as $ijk \ldots$ is inevitable.

As the distance function is a sum of squares, this geometry is of the flat or Euclidean type. All simple types of geometry become flat in the

smallest regions; thus we may attempt to construct them by piecing together small flat regions, so as to form a uniform whole; and in this way Riemann has formed (in extension of Gauss) the elliptic and hyperbolic geometries by the method of the Calculus of Variations. Or we may consider the region around every material nucleus to be affected by its presence, and so to be non-uniform; and we may thus with Einstein propound the problem whether universal gravitation may be adequately or fruitfully represented by local deformation of space, instantaneously re-established whenever disturbance arises through convection of material nuclei, rather than by a field of stress which, on account of such very rapid restitution, could not be absorbed into the electrodynamic scheme of relativity.

Finally, the results of Mr. Johnston's note may be summarised. The "four-potential," F, G, H, ϕ, recognised by Minkowski,* may be expressed by the symbol U, which represents $\mathrm{F}i+\mathrm{G}j+\mathrm{H}k+\mathrm{W}o$; and the usual spacial operator $i\,\partial/\partial x+j\,\partial/\partial y+k\,\partial/\partial z+o\,\partial/\partial w$ is represented by ∇_1. Then when $\sqrt{-1}\,ct$ is put for w and $\sqrt{-1}\,\phi/c$ for W, the expression for $\nabla_1\mathrm{U}$ determines the conjoint electric and magnetic fields in terms of his notation by (4), viz.,

$$c\nabla_1\mathrm{U} = jk\mathrm{L}+ki\mathrm{M}+ij\mathrm{N}-\sqrt{-1}\,(io\mathrm{X}+jo\mathrm{Y}+ko\mathrm{Z}),$$

where XYZ and LMN are the electric and magnetic intensities, the scalar part (3) vanishing in virtue of the electric field being devoid of convergence except at its sources. His expression for $\nabla_1\nabla_1\mathrm{U}$ consists of eight terms as follows:—

$$\nabla_1{}^2\mathrm{U} = \sqrt{-1}o\mathrm{A}-i\mathrm{C}-j\mathrm{C}'-k\mathrm{C}''-\sqrt{-1}(jko\mathrm{D}+kio\mathrm{D}'+ijo\mathrm{D}'')+ijk\mathrm{B}\,;$$

and its vanishing involves that of A, B, C, C′, C″, D, D′, D″ separately, giving the eight equations of the electrodynamic field in free space in the order in which they are written in his note.†

Thus all is expressed in terms of two vectors, U and ∇_1, in the unitary scheme typified by $i^2 = -1$ and $ij = -ji$, but with no other restriction such as the $ijk = -1$ of quaternions.

* The procedure of Minkowski seems to have been, having identified electrodynamic relativity with invariance of the system as regards position in the fourfold continuum, to group and identify the physical quantities of the Maxwellian field as components of various 4-vectors and 6-vectors, constructed independently from the single fourfold vector potential so that their invariance is analytically recognisable. This gave rise to an algebra of the various types of vectors. The procedure in the present calculus is the reverse. The system of invariants natural to a four-dimensional flat continuum are immediately manifest in Mr. Johnston's application of Clifford's calculus; and the totality of them are identified precisely with the vectors of the electrodynamic scheme of Maxwell, with which they are co-extensive. The formal scheme of physical nature as regards electrodynamic and radiational phenomena is thus concomitant with the geometry of a single vector function in four Euclidean dimensions of space and time.

† These equations are set out *infra* on p. 345.

It is only in so far as a hyper-quaternion calculus can be framed, operating directly with hyper-vectors, such as ∇_1 and U, without any relation to a special unitary system of reference such as $ijk\ldots$, that a method of direct relativity can be said to assert itself. Failing such a calculus, it is necessary to fall back on the indirect method, which cannot help using $ijk\ldots$, but bases the relativity on the proposition that there is nothing to distinguish one such system from any other; the conclusion is, in substance, that if we could transcend the modes of representation of the physical world that are open to us, we should discover direct relativity. In this continual effort to transcend, the method of Variations on Riemann's foundation has been a substantial feature; the symbolic calculus now under consideration seems to constitute another direct help to hyper-spacial synthesis.

Here the sole function of the quantity c is to make cdt of the same physical dimensions as dx: by change of unit it may be reduced to unity. But the alternative mode of expression,

$$\nabla_1^2 \mathrm{U} = 0, \text{ involving the vanishing of } \nabla_1^2\mathrm{F},\ \nabla_1^2\mathrm{G},\ \nabla_1^2\mathrm{H}, \text{ and } \nabla_1^2\phi,$$

implies on the ordinary theory of space and time that F, G, H, ϕ, and therefore the whole disturbance, travel out in three dimensions of space from located point-sources with definite velocity c, which thus enters as the velocity of radiation; while the permanence of the sources has to be adapted to the scheme by FitzGerald-Lorentz deformations of their collocation.

Thus a scheme of the kind above specified, however symmetrical and concise as a representation of a field of activity, seems not to be so effective in the representation of its sources. An electron or an atom is a singularity permanent though mobile in space; that quality is essential to the idea of matter. Whereas in the continuum of four dimensions the electron is represented, not by a singular point, but by a curved line (that of Minkowski) with a condition to be satisfied all along it. The element of time is in this respect an obtrusion; the compact mobile singular point in three dimensions of space seems to be the natural expression of permanent unchanging existence.

(*Added October* 20, 1919.)

Analysis of Possibilities in Geometric Algebras.—Restriction is at first made to three dimensions: this will illustrate the general case, though new features will there come in through the higher products. The position-vector R from the origin to the point xyz, which satisfies the criterion of a vector being an entity independent of orientation of co-ordinates, is represented, at first tentatively, by the bilinear form $ix+jy+kz$. When the system of axes of co-ordinates is altered xyz become $x'y'z'$: and, when we

write the usual geometric equations of transformation, and so introduce a new trihedron, $i'j'k'$, related in the contravariant manner to ijk,

$$\begin{aligned} x &= l_1x'+m_1y'+n_1z', & i' &= l_1i+l_2j+l_3k,\\ y &= l_2x'+m_2y'+n_2z', & j' &= m_1i+m_2j+m_3k,\\ z &= l_3x'+m_3y'+n_3z', & k' &= n_1i+n_2j+n_3k, \end{aligned}$$

then the position-vector R is also expressed by R′, equal to $i'x'+j'y'+k'z'$.

By the known relations of Euclidean geometry, it is now easy to verify that the unitary system ijk does, in fact, remain invariant as to type, except that the product ijk changes sign when the system of axes undergoes perversion, represented symmetrically by reversal of sign of all the units. Thus, for the group of rotations without perversion, $i^2+j^2+k^2$ is invariant, and the product ijk as above. But a binary product $i'j'$ is not expressible in terms of like products ij, jk, ki unless $i^2=j^2=k^2$; and then, provided further $ij=-ji$, ..., ..., it will have expression of type

$$i'j' = n_1jk+n_2ki+n_3ij,$$

here with no change of sign for symmetrical perversion. Thus the unitary relations $ij=-ji$, ..., ..., persist on transformation, so are invariant. Also (jk, ki, ij) is cogredient with (i, j, k) so that the quaternion simplification is suggested which avoids two kinds of vectors, making them identical by postulating that ijk is a scalar constant.

The square of $\mathrm{R}_1-\mathrm{R}_2$ is now $i^2\{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2\}$. As it must be invariant for change of trihedral axes, it is necessary that i^2 be the same for all directions of the component vector i; it must therefore be a scalar, say A. This involves and secures that the square of any vector is a scalar, say is the square of its tensor; and the vanishing of the tensor involves that of all the components of a real vector.

Thus the unitary system has been restricted to the form

$$i^2=j^2=k^2=\mathrm{A}, \qquad ijk=\mathrm{B},$$
$$ij=-ji, \qquad jk=-kj, \qquad ki=-ik,$$

involving three derived relations of type $\mathrm{A}ij=\mathrm{B}k$.

It would seem at first that, to avoid complexities, the symmetric entity B must be scalar as well as A, and then we have $\mathrm{B}^2=-\mathrm{A}^3$, so that both can be real only when A is negative. But this holds, as will appear, rather for quaternions than for pure vectors.

We can now further simplify the symmetric group of units by multiplying them all by the same scalar, viz., by any quantity of ordinary algebra, real or complex. If this multiplier is $\sqrt{-\mathrm{A}}$, the plan of a spacial algebra is reduced to the form

$$i^2=j^2=k^2=-1, \qquad ij=-ji, \ldots, \ldots, \qquad ijk=-1 \text{ or } +1.$$

The first alternative is the unitary scheme belonging to quaternions; the second alternative is the same scheme perverted symmetrically by change of sign of all the units.

A scheme with $i^2 = j^2 = k^2 = +1$ cannot be obtained for the flat space except by multiplying each unit by $\sqrt{-1}$, which involves every component of every vector being a pure imaginary.

Perversion of a space (in the sense of Listing and Maxwell) is transformation into a new space, the mirror image of the former; thus change from i to $-i$ is equivalent to a perversion. Two perversions in succession are equivalent to a rotation without intrinsic change; thus the quaternion unitary system is not altered by change of sign of both i and j.

The notation of quaternions, so far as regards pure vectors, was introduced into electrodynamic relations by Maxwell as a convenient means of exhibiting their independence of all special systems of co-ordinates, but not, at that stage, of all assignable systems of translatory convection. The earlier formulæ of direct attraction evolved by Ampère, Grassmann, and others were constructed on scalar ideas; but, as they implied instantaneous transmission, they were not affected by any finite motion imparted to the system, whether uniform or not. The effort to express the electrodynamic and other relations in terms of vectors independent of frames of reference in space, and also independent of uniform convection in space to which the formulæ of material dynamics and perhaps gravitation already give no response, is in the main the modern analytical problem of relativity.

Limitation of Relativity to the Free Æther.—The invariant scheme for an electrodynamic field does not constitute complete relativity. It is merely concerned with the changing field itself, of which the abiding discrete entities of nature are the sources. It is the relations between the latter that constitute the facts of existence: their own permanence throughout change of relations is implied and must be verified. Now the field of existence for this purpose is not that of x, y, z, t, but that of x, y, z alone: for matter and other permanent foundations exist not in time but irrespective of it. An electron is a singularity in space, whose essential qualities are independent of time. This may be claimed to be a true discrimination, which must prevent time being treated after the manner of another dimension of space. Events occur both as regards space and time, but the entities that substantiate these events, being the permanent constants in their mathematical representation, have no relation to time: phenomena are in space and time, and relate to substance, but that (unless we take matter to be a fleeting show) is the essential substratum into which time does not enter.

Thus, for example, for the equation $\nabla_1^2\phi = 0$, the basic type of solution

$(x^2+y^2+z^2)^{-\frac{1}{2}}f\{ct-(x^2+y^2+z^2)^{\frac{1}{2}}\}$, which in three dimensions expresses the fact of clean propagation without leaving a trail, such as exists for three dimensions but not for two, has in the four dimensions for its singularity or source by which it enters into the manifold, the line which is the axis of t. A point singularity would now belong to a point of space and to an instant of time: the field of which it is the nucleus, which is traceable back to it, would be regarded as introduced into the system through that point-source at a definite time. Such an instantaneous nucleus would be entitled to the designation of a *miracle*, as distinct from the usual permanent nucleus which would be a mobile *point of matter* or other abiding source. Such instantaneous nuclei may be amenable to treatment analytically by the methods of Green's memoir on potentials of ellipsoids in n dimensions: the simplest type of ϕ, that which corresponds to the ordinary potential, is

$$\mathrm{A}\{(x-p)^2+(y-q)^2+(z-r)^2-c^2(t-\tau)^2\}^{-1},$$

having a singularity which at the instant τ is at the point (p, q, r) and as time changes expands into an infinite source spread over a spherical surface, and thus is impracticable.

Ideas of propagation are, however, excluded in the fourfold space-time continuum, which presents a static map of the entire historical world-process at one glance. Though quantitative permanence as regards the inertial measure of matter evaporates, physical configurational permanence of the nuclei which constitute it would perhaps remain, while inertia is transferred hypothetically to energy. It is of interest in this connection to recall that when a permanent isolated material system is moving through space, and radiating away its thermal energy, while its total energy and momentum and inertia continually diminish, yet its velocity remains constant.* These results follow on the usual electrodynamic principles; it is not a case of relatively transcending them, for the effects are of the first order in v/c: the velocity contemplated is thus velocity with respect to the æther. From the other point of view, such constancy of velocity is demanded by the relativity scheme as established for electrodynamics; for, if the velocity were retarded by the reaction of radiation, the system could not remain invariant when referred to a space-time frame moving along with itself. [This latter statement is thus the criterion on relativity principles that replaces the Newtonian first law of motion.†]

Time contrasted with Space in a Permanent World.—Analytically, varieties of geometric algebras may be obtained by altering the values of the squares of

* See Poynting's 'Scientific Papers,' appendix added of date 1918.

† *Cf.* a cognate discussion in 'Proc. American National Academy,' 1917.

the various units between the alternatives $+1$ or -1, or even by putting some of them equal to zero as in Grassmann's polar elements. The polar relations of type $ij=-ji$ must not be tampered with. For if any of them were changed to the type $ij=+ji$ the square of the distance of two points would no longer be invariant: we would have an algebra appropriate to some type of manifold, but that would no longer be a flat or Euclidean space whether real or imaginary.

The algebra of Hamiltonian quaternions of type $wr+xi+yj+zk$, in which one of the units r is the real quantity $+1$, is not a four-dimensional geometric algebra in the present sense, but is rather an algebra of two dimensions, viz., ijk limited by $ijk=-1$, operating with another isolated dimension r of a different kind. To obtain in it a distance invariant in the four dimensions one must form the product with another quaternion, complementary to it as regards w, namely $wr-ix-jy-kz$, this product being $w^2+x^2+y^2+z^2$. And similarly the operator $\left(\frac{\partial}{\partial w}\right)^2+\left(\frac{\partial}{\partial x}\right)^2+\left(\frac{\partial}{\partial y}\right)^2+\left(\frac{\partial}{\partial z}\right)^2$ is invariant only in this limited sense.

Thus the usual quaternion system can be developed: but it would appear to be rather an algebra of interaction between two types of vectors, one in three limited dimensions ijk, the other in the dimension r, these vectors combining as wholes according to the ordinary algebra of scalar quantities.

In a quaternary algebra, $ijko$, of direct geometric type, we cannot have $\left(\frac{\partial}{\partial x}\right)^2+\left(\frac{\partial}{\partial y}\right)^2+\left(\frac{\partial}{\partial z}\right)^2$ invariant, nor $x^2+y^2+z^2$, unless it is a ternary geometric algebra combining with a unitary one o. It then breaks up into an algebra of space and an algebra of time, the two combining by some scheme of rules such as those of the algebra of scalars. Except in this special case there cannot be permanent particles of matter or electrons belonging to the system (except as nuclei purely configurational) any more than there can be permanent spacial distances not involving time.

Electrodynamic relativity seems to be secured for the field in free æther by sacrificing them both: particles and electrons and space are resolved, so that only Minkowski's singular curves or "world lines" and their intersections—in a later form the intersections alone—survive. The material world vanishes in order that relativity may survive. It can thus be held to be more direct and practical, in relation to the complete expression of the order of nature, to persist so far as possible with the procedure expressed in terms of matter and motion, in the Newtonian manner; this requires and involves slight deformation of the material structure of measuring systems, and thereby is quite competent to secure, in relation to matter which alone can be

observed, the observed relativity that is otherwise interpretable by merging time with universal space into one interlocked manifold.*

It may be noted that schemes in hyperspace and hypertime might be constructed in which time would be endowed with more than one dimension: there could be an algebra of time vectors operating with space vectors according to some scheme of laws, the simplest case being the mutual independence of time and space as above that goes with the laws of ordinary algebra.

Thus we appear to arrive again from the other side at the position previously asserted. Relations from their very nature must be between entities that possess features of permanence when the point of view (scheme of co-ordinates) is changed. Mutuality or relativity will be futile except as mere symbolism if these entities are not maintained. To establish the possibility of permanent material or electric systems it seems to be necessary to treat space by an algebra of its own distinct from time, in the manner originally enunciated summarily as a postulate, the cause of much misunderstanding at the time, by Newton in the 'Principia.'

Relations, which are the subject matter of knowledge, are changing in time: but there must be underlying things that are related, defined as of permanent type, even though in other respects known only through these relations, that is relatively. This duality, relations and entities related, appears to be unavoidable; the possibility of knowledge implies both.

The apparent present is merely a boundary between past history and future evolution. The past would thus be the real present that is with us, and memory the purveyor of the materials of systematic knowlege.†

But, whatever be the critical obstacles, the problem of probable interaction between gravity and electrodynamic fields, including rays of light, of course remains urgent; and if such connection is actually detected by the various astronomical determinations now in progress, data such as hitherto have been entirely non-existent will have been supplied for an attack on this deep-seated question.‡

* This holds for the older experimental evidence of Michelson, Rayleigh, Brace, Fitzgerald and Trouton; the more recent astronomical evidence, involving the interplanetary spaces, cannot be so incorporated.

† It may be noted that the late Lord Rayleigh, in a Presidential Address to the Society for Psychical Research last April, hinted at his difficulty in understanding how past and future could possibly be related merely as North and South.

‡ This was written before it became known that the Greenwich and Cambridge astronomers, in their recent eclipse expeditions, had confirmed Einstein's prediction for the amount of the deflection of a ray of light by the influence of the Sun. It must be recognized that the theory has come to stay in some form or other. Its main implication, of instantaneous propagation of change in the constitution of space, seems to

The relation that ijk is a scalar, say -1, is not a necessary part of the geometric algebra: it is introduced in the mixed scalar and vector of quaternions in order to limit all the derived vectors to one type, and so get direct geometrical interpretation. But in the quaternary algebra, to restrict $ijko$ to a scalar value would retain the algebraic geometry, while limiting its powers of representation; thus, for example, it would obliterate the expression of the electrodynamic scheme by combining its eight equations in pairs into only four. It would be really a ternary algebra with $\pm o$ for the value of the product ijk.

The spacial algebra may in fact be elaborated in various ways. In the general case of n dimensions the distance of two points is invariant for algebraic geometries that involve all the relations $i^2=-1, ij=-ji$; and this appears to be irrespective of how their unitary scheme may be completed by assigning laws and interpretations for triple and higher products.

May we say that all such completed systems represent flat or Euclidean spaces, and so are equivalent at bottom to the simplest single one that can be constructed, but that they differ as regards richness of detail? Viewed from the geometrical side the answer seems to be definite; a geometry of lines and distances appears to be quite feasible by itself on the basis of translation and rotation, as in Euclid, without advancing at all into special interpretations of the higher products, which could represent, through various schemes, quantities such as areas and volumes and their orientations.

As any invariant expression formed by multiplication of vectors is homogeneous in the units, though the reduction by substitution of -1 for i^2, etc., makes it apparently not so, alteration of all the units by multiplying by the same scalar does not change the expression. It is therefore invariant as a whole, but not divisible into independent invariant parts; except that the parts of odd and even orders in the units are separately invariant, as with Clifford. In the expression for $\nabla_1{}^2\mathrm{U}$ above, the two parts, one homogeneous of the first degree and the other of the third degree in $ijko$, cannot be other than conjugate components of a single generalised eightfold vector.

This process of evolving something completely different, an undirected scalar, from the interactions of directed vectors, was the startling innovation of the symbolic algebraic analysis. It perhaps had its ultimate suggestion, in much simpler form, in the imaginary of ordinary algebra and the Argand

be avoidable only on a psychological point of view which would assert that a portion of space is existent only while attention is concentrated on it. It can be managed, however, by including the varying space in a uniform space of higher dimensions, just as the deformation of a two-dimensional surface can be visualized as a whole in uniform space of three dimensions; *cf. infra*, p. 353.

diagram; this however involved a real and a complex or symbolic unit, whereas in a vector scheme, self-symmetrical, all units must be alike and symbolic.* It can be developed on a purely logical unitary basis as de Morgan's "double algebra," whereas spacial geometry must run its algebra three abreast. Without the mode of representation as spacial threefold continuity, inherent in the mind, as the intuitive guide in an algebraic process, a calculus so remote from the usual chain of ideas of algebraic combination would have had small chance of emerging into light.

On Imaginary Space.—It seems legitimate, as above, to take each component of a general vector to be itself of the most general type of simple unidimensional algebra, namely, a complex quantity. But to do this with one of the components of the space vector is, in the usual terminology, to make that dimension of space imaginary. This would more consistently, in algebra, be effected rather by modifying the scheme of unit vectors that define the manifold, keeping the co-ordinate scalar multipliers real. Thus for the usual scheme of relativity, in which the fourth dimension, that of time, has to be a pure imaginary, instead of writing $\mathrm{R} = ix+jy+kz+(c\sqrt{-1})\,to$ we could express it as $\mathrm{R} = ix+jy+kz+(ct)(\sqrt{-1}\,.\,o)$; and similarly we would write $\mathrm{U} = i\mathrm{F}+j\mathrm{G}+k\mathrm{H}+(\sqrt{-1}\,.\,o)\,c^{-1}\phi$; pointing (as Mr. Johnston has remarked to me) to the introduction of a unitary system $ijko'$, the latter o' taking the place of $o\sqrt{-1}$, defined by the relations

$$i^2 = j^2 = k^2 = -1, \quad o'^2 = +1, \quad ij = -ji, \ldots, \ldots, \quad io' = -o'i, \ldots, \ldots.$$

This contrasts with the quaternion group of four, scalar *plus* vector, in which o is merely the real numerical unit.

The analogy of the analytical manifold called an imaginary space to real space is so imperfect for practical purposes as to be almost fictitious when the space is flat. To realise it, generalisation is necessary into curved space such as that of a spherical surface: for example, the region within a closed ellipsoid has its boundary changed to one represented by an open hyperboloid, in flat space. As the new auxiliary manifold of relativity has nothing to do, except symbolically, with ordinary space, whether Euclidean or not, it is a more self-contained procedure to construct it independently on a basis of its own by an associative algebra of type $ijko'$.

We recapitulate the electrodynamic scheme on this basis, partly in order to emphasise another principle which it aptly illustrates.

The manifold in which the position-vector is represented by

$$\mathrm{R} = ix+jy+kz+o'ct$$

* See the discussion in Clifford's 'Math. Papers,' "On Biquaternions," especially the posthumous 'Further Note,' p. 385.

implies $(R_1 - R_2)^2$ invariant, that is $(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2-c^2(t_1-t_2)^2$ invariant; it is thereby self-consistent and determined, in a form that can be visualised and so verified symbolically as an imaginary Euclidean space.

The invariant entity $\nabla_1 U$, where U is $iF+jG+kH-o'c^{-1}\phi$ and ∇_1 represents $i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}+o'\frac{\partial}{\partial(ct)}$, is expressed as

$$\nabla_1 U = jk\alpha + ki\beta + ij\gamma + c^{-1}(io'P + jo'Q + ko'R);$$

for its value when set out at length in this way involves the electric force PQR and the magnetic force $\alpha\beta\gamma$ of electrodynamics of free space, as expressed in terms of their partial potentials* $FGH\phi$, and also involves the electrodynamic relation of convergence $\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}+\frac{\partial H}{\partial z}+c^{-2}\frac{\partial \phi}{\partial t}=0$.

The repetition of the same operation gives

$$\nabla_1^2 U = ijk\left(\frac{\partial\alpha}{\partial x}+\frac{\partial\beta}{\partial y}+\frac{\partial\gamma}{\partial z}\right)+ijo'c^{-1}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}+\frac{\partial\gamma}{\partial t}\right)+\ldots+\ldots$$
$$-o'c^{-1}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)+i\left(\frac{\partial\gamma}{\partial y}-\frac{\partial\beta}{\partial z}-c^{-2}\frac{\partial P}{\partial t}\right)+\ldots+\ldots.$$

Thus $\nabla_1^2 U$ is expressed in terms of two partial vectors, with units $ikjo'$ and their ternary products respectively, so named because they are only components of a complete complex, which in its entirety, and not as regards its separate components, enjoys the invariant property. The single equation $\nabla_1^2 U = 0$ sums up the relations of the electrodynamic field in free space.†

[*November* 7, 1919.—The equation above, which expresses the vanishing of the scalar part of $\nabla_1 U$, viz.,

$$\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}+\frac{\partial H}{\partial z}+\frac{1}{c^2}\frac{\partial \phi}{\partial t}=0$$

is equivalent to

$$-\nabla_1^2\phi = \frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}.$$

* The electromagnetic units of Maxwell's 'Treatise' are here employed, in which curl $(F, G, H) = (\alpha, \beta, \gamma)$ and $P = -\partial F/\partial t - \partial\phi/\partial x$. In the Gauss-Helmholtz mixed electrostatic-magnetostatic system employed in Mr. Johnston's note, the first relation would be c curl $(F, G, H) = (\alpha, \beta, \gamma)$, and c occurs more symmetrically in the field equations. A rational way of getting rid of this embarrassing confusion between the units of practical science and the symmetrical Hertz-Heaviside modification is to eliminate c altogether in the mathematical analysis by making it unity. In the final results it could be restored by inserting in each term as a factor such power of c as is needed for uniformity of physical dimensions in the unitary system that is preferred.

† Observe that two of these eight relations are involved in the other six: this consistency is a test of the validity of the symbolic algebra.

The left side exhibits an invariant operator acting on a component of the invariant vector potential U, which is not itself invariant. Also the potential ϕ is not cleanly propagated except in regions where the right side of the equation vanishes. As it has been known from the first* that the distribution of charge is invariant, it follows that the expression on the right is not the measure of the charge though it vanishes in free space where there is no charge: that is because the element of volume changes on transformation as *infra*.

The symbolic calculus here under consideration adapts itself readily to the electrodynamic formulation for a space pervaded by electric flux and electric density expressed (after Lorentz) by continuous functions: and it is interesting to observe the successive stages of restriction. The invariance of the charge also emerges in a different way. Instead of $\nabla_1{}^2\mathrm{U}$ vanishing, we assert an equation, which is invariant, of form

$$-\frac{1}{4\pi}\nabla_1{}^2\mathrm{U} = iu_1 + jv_1 + kw_1 + o'c\rho + \text{zero},$$

the zero indicating that the vector-components of other type are to be made to vanish as they do identically. In other words,

$$\nabla_1{}^2(\mathrm{F}, \mathrm{G}, \mathrm{H}) = -4\pi(u_1, v_1, w_1), \qquad \nabla_1{}^2\phi = -4\pi c^2\rho.$$

We may regard u_1, v_1, w_1, ρ as quantities to be interpreted. If we operate a third time with the invariant operator ∇_1, on this simple vector $-\mathrm{U}/4\pi$, the scalar part of the result is

$$-\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}+\frac{\partial w_1}{\partial z}-\frac{\partial \rho}{\partial t}\right).$$

This quantity is therefore invariant for all frames of reference.† That being so, it is permissible to impose the physical restriction that this invariant value shall be zero. Then it asserts that (u_1, v_1, w_1) is some sort of circuital flow, or briefly of streaming, to which is added a convection of an electric density ρ. If it be further postulated that there is no such streaming, but only convection of the electric density, we have

$$(u_1, v_1, w_1) = \rho(\dot{x}, \dot{y}, \dot{z}).$$

* See 'Æther and Matter,' § 112.

† Other invariant vectors may be noted. Thus the mechanical force on the medium is expressed by three of a set of four components of the product $-\frac{1}{4\pi}\nabla_1{}^2\mathrm{U}\,.\,\nabla_1\mathrm{U}$; the Maxwellian stress system by components of the product of $\nabla_1\mathrm{U}$ by a conjugate vector derived by changing signs of some of the units $ijko'$, and so on.

Thus now

$$-\frac{1}{4\pi}\nabla_1{}^2\mathrm{U} = i\rho\dot{x}+j\rho\dot{y}+k\rho\dot{z}+o'c\rho\,;$$

and its square with changed sign is

$$\frac{\rho^2}{dt^2}(dx^2+dy^2+dz^2-c^2dt^2).$$

Now the second factor of this expression is the invariant expressing the flatness of the four-dimensional continuum; hence the other factor, and so ρ/dt, must also be invariant. Also if $d\tau$ is an element of ordinary volume $d\tau dt$ is invariant, being the element of four-dimensional extension. Therefore $\rho d\tau$ is invariant,* which expresses persistence of value of electric charge when the reference is changed from one frame to another. It depends expressly on the postulate that all electric current is convection of charge. The inclusion of polarisation-charges would require special treatment].

In passing, it may be noted that $-\mathrm{U}^2 = \mathrm{F}^2+\mathrm{G}^2+\mathrm{H}^2-c^{-2}\phi^2$, so is scalar and invariant for change of axial system. Also, the scalar component of $(\nabla_1\mathrm{U})^2$ represents the volume-density of the Lagrangian function or of the kinetic potential in the four-dimensional space, being thus invariant itself, and securing by the Hamiltonian principle general dynamical invariance. Energy density is the scalar part of the product of two conjugates $\nabla_1\mathrm{U}$, $\nabla_1\mathrm{U}'$, and is not completely invariant.

But it is to be noted that, if the unitary scheme is restricted, after the manner that the quaternion scheme is restricted, by making the product of $ijko'$ a scalar, the two partial vectors in $\nabla_1{}^2\mathrm{U}$ lose their separateness of type, add together, and the complete electrodynamic formulation will be lost. The restricted system has been made too narrow, though still adequate for some other purposes of description in the manifold.

A set of units, with proper symbolic laws of combination, is what extends the scope of pure algebra from relations of linear measures to those belonging to more complex constructs in the continuum; but, when the scheme of their laws of combination is restricted, they may continue to be adequate for the simpler forms of the space relation, while they lose part of the power of expressing wider constructs through their combinations, and thus representing the interactions of physical phenomena arising in the space. While the quaternion restriction $ijk = -1$ secures agreement with spacial intuition by involving only one kind of simple vector, the power of direct representation may thereby as here be much restricted.

* *Cf.* E. Cunningham's exposition, after Minkowski, 'Principles of Relativity,' p. 105.

On the other hand, the products of second, third, and higher orders in a quaternary algebra *ijko* are outside geometrical intuition; but Mr. Johnston has established with logical precision (as I understand) the validity of the associative principle in Clifford's algebra, and has also shown that the algebra involves, by introduction of twelve new auxiliary units, constructs of quaternion type, capable of including and representing Grassmann's various species of geometrical products.

It may be recalled that the Cartesian geometry involves already a symbolic algebra, that of the unit -1 which reverses a vector. The algebra of ordinary complex imaginaries, as elucidated geometrically, is a binary one involving two symbolic units, -1, or say α, and $\sqrt{-1}$, or say β, and the real unit γ, or $+1$, with the laws of combination $\alpha^2 = \gamma^2$, $\beta^2 = \alpha\gamma = \gamma\alpha$, $\alpha\beta = \beta\alpha$, $\beta\gamma = \gamma\beta$. An algebra of polar units ijk ... can be superposed on this one, as remarked above.

As already noted, the fact that the continuum *ijko′* is of only four dimensions does not preclude the formation in it of a generalised vector such as $\nabla_1{}^2 U$, having eight independent components. A parallel conception is the line-geometry, imagined also independently by Plücker, in which each line has four independent co-ordinates, or six symmetrical ones (the components in fact of Clifford's rotor) with two relations between them, though the space in which this geometry has its play has only three dimensions. As two lines (or forces) are not identical unless their co-ordinates all agree, so two eight-vectors cannot be identical, or the vector expressing their difference cannot vanish, unless all the components of the latter vanish separately. Or, again, there is the fourfold geometry in ordinary space, in which the spherical surface is the unit instead of the point, with combinations of its own: and so more generally. So also the vectors of the electrodynamic field have modes of combinations of their own, to which the scheme above presented is completely adapted.

In a real flat geometry represented by *ijko*, dealing with pure vectors without the scalar part characteristic of quaternions, the square of the tensor of any generalised vector is a scalar represented by a sum of squares; thus the vanishing of the tensor—analogous to distance—involves that of the vector if it is a real one. But in an imaginary geometry such as *ijko′*, some of the squares are affected by the negative sign, and the vanishing of the tensor does not carry this result. Many theorems of determinacy which are valid for the real space fail for the imaginary one.

The Spacialised Relativity: its Limited Range.—The converse mode of argument seems to lead directly to unexpected limitation of purely relative theories. For we can examine the degree of generality possible in a field of physical

activity, limited by only two characteristics: (i) effects are to be simply propagated across the æther, without leaving any trail, and all with the same standard speed when they do not travel instantaneously;* (ii) uniform translational convection of the whole field of activity through the æther is not to produce any recognisable effect in the internal relations of that field.

The first characteristic requires that the distributions of scalar quantity specifying the field, say the functions F, G, ..., should all satisfy the equation of simple propagation, which for three dimensions of space and one of time is of the type $\nabla_1^2 F = 0$. The second characteristic requires that in the manifold x, y, z, ct, these scalar quantities should be the components of a physical vector, say of U, equal to $iF + jG + kH - o'c^{-1}\phi$; in this form they are virtually limited in number to four, the analysis for other conceivable types such as a six-vector, or for more than one vector U, being put aside as foreign to the actual physical type and impracticably complex. The operation ∇_1 or $i\,\partial/\partial x + j\,\partial/\partial y + k\,\partial/\partial z + o'\,\partial/\partial(ct)$ is an invariant vector analogous to a position-vector, because $\partial/\partial x$, $\partial/\partial y$, ..., are transformed and combine according to the same type as x, y, ...; and it is the unique simple vectorial differential operator that is available. This equation $\nabla_1^2 U = 0$, which is the complete expression of simple propagation with uniform velocity, has also been shown, when constructed in two stages, to be the expression of Maxwell's electrodynamic equations; for they are formulated in terms of the vector components of $\nabla_1 U$ as variables, these components representing the electric and magnetic forces of the field. For this purpose the scalar part of $\nabla_1 U$ has been equated to zero in advance, being thus an equation of continuity, expressing that the convergence of the vector U vanishes. In any case, if that were not done, the vanishing of $\nabla_1^2 U$ would in itself involve the vanishing of the gradient of this scalar convergence, and so make its value constant and therefore null. The two criteria, (i) and (ii), thus by themselves alone severely restrict possibilities, in an isotropic medium such as free æther with only one characteristic speed, to Maxwell's scheme of equations of the electrodynamic field.†

Expressed in Sylvester's terminology‡ the invariance of the binary linear form $ix + jy + kz$, which is made fundamental at the beginning of this note,

* If there were two standard speeds, as there are for rotational and compressional waves in an elastic solid medium, their interplay would determine velocities of convection absolutely, so relativity would be excluded.

† This conclusion has been reached, as I now find, by Ph. Frank in 'Ann. der Physik,' vol. 35, p. 599 (1911), who ascribes it in more general form to H. Bateman, 'Proc. London Math. Soc.,' 1910. His analysis involves, after Sommerfeld, recognition of the products of the *Ausdehnungslehre*.

‡ Salmon, 'Higher Algebra,' §127.

requires that where (x, y, z) is transformed by any linear substitution (i, j, k) must be transformed by the contragredient substitution: this involves that if (x, y, z) is transformable by a rotation of a rigid geometrical system (i, j, k) is transformable also by a rotation, but one belonging to the conjugate (perverted) group. The characteristics of the polar units necessarily persist in this transformation; for it is already settled that on account of the invariance of distance the scheme can represent spacial relations.

The following treatment of the last discussed problem is a direct example. The transformation for which wave-motions with velocity c persist are those for which ∇_1^2 operating on U is invariant. Now $x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+w\frac{\partial}{\partial w}$, where w is ιct, is an invariant bilinear form; hence $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial w}\right)$ are contragredient to (x, y, z, w). Hence invariance of the operator ∇_1^2 involves invariance of $x^2+y^2+z^2-c^2t^2$; that is, it obtains only under the Lorentz transformation, a result reached by Voigt by direct algebra a long time ago.

If the changes were very slow, or c very great, the field would be simply electrostatic, in which an electric nucleus repels another of the same kind. That is because the energy of the field is of static elastic type: if it were of the steady kinetic type there would be mutual attraction, as in gravitation.

It is because Mr. Johnston's rediscovery and adaptation of Clifford's calculus, though quaternary, is wider and more symmetrical than quaternions, that it is able directly to grasp and consolidate so much in the relations of physical analysis, though without the intuitive geometrical interpretation that quaternions enjoy. At the same time, like quaternions, it has the advantage of being condensed into one algebra involving unique modes of addition and multiplication of single variables: the separately defined products of other vector analyses arise in it compactly as components of the single product with which alone it has to be concerned.

I have been indebted to Mr. Johnston for criticism and corrections of this note. When it was written Prof. W. K. Clifford's papers relevant to the subject were not at hand. It seems desirable to add brief references to his work. He propounded his "Biquaternions" in 1873 as the algebraic operators which change the general position of a rigid body. Some time after, he became interested in general linear algebras with polar units, and recognised their relation to that special calculus. In 1878, the year before he died, he introduced a memoir* entitled "Applications of Grassmann's Extensive Algebra," published in the first volume of the 'American Journal of

* No. XXX in 'Math. Papers.'

Mathematics,' by the following pathetic sentences. "I propose to communicate in a brief form some applications of Grassmann's theory which it seems unlikely that I shall find time to set forth at proper length, though I have waited long for it. Until recently I was unacquainted with the 'Ausdehnungslehre' . . . . I may, perhaps, therefore be permitted to express my profound admiration of that extraordinary work, and my conviction that its principles will exercise a vast influence upon the future of mathematical science." The paper determines, as he states, the place of quaternions and biquaternions "in the more extended system, thereby *explaining* the laws of those algebras in terms of simple laws;" it generalises them to higher dimensions and proves "that the algebra thus obtained is always a compound of quaternion algebras that do not interfere with one another." There is also a cognate posthumous fragment* "On the Classification of Geometric Algebras." The prefatory introduction by H. J. S. Smith, pp. lv–lxvii, gives a sketch of the work, in which it is held, apparently with justice, that in bringing together Hamilton's and Grassmann's work Clifford had transcended both their points of view.

In the present note the algebra of Clifford is connected with spacial relations on the basis of the single criterion that it makes the square of the Euclidean distance of two points invariant for all transformations, for the reason that it is the square of the difference of their position-vectors. In quaternions distance enters in quite a different manner as a tensor. There are in Clifford's work indications, deserving fuller elucidation, how the flat or Euclidean geometry of four dimensions becomes elliptic geometry of three dimensions, with its more complex notion of distance, when the polar units are transformed to quaternion units in this way.

The apparent paradox in Clifford's algebra, on which H. J. S. Smith has put some stress, that although $\iota_1\iota_2 = -\iota_2\iota_1$, yet $\iota_1^2 = -1$ and not $= 0$, is also present in quaternions, from which not improbably it was introduced; but really ι_1 and ι_2 are definite discrete entities, and the suggested transition to a limit when they are equal is not relevant, except within the narrower bounds of Grassmann's geometrical calculus of distances.

Clifford definitely separates the quaternary algebra based on units ι_1, ι_2, ι_3, ι_4, defined by polar multiplication, together with $\iota^2 = -1$ for each unit, into two algebras, one concerned with products of even order, the other with products of odd order. This is done by introducing a new unit, w, equal to the product of $\iota_1, \iota_2, \iota_3, \iota_4$; the unitary system thus becomes $\iota_1, \iota_2, \iota_3, w$, which is still polar, the only change of type being that $w^2 = +1$ instead of -1, but

* No. XLIII in 'Math. Papers.'

at the same time the orders of product terms are much reduced. For the algebra of even products, he recognises that this system is identical with his special calculus of Biquaternions. In the present physical application, this separation into two simpler algebras is however excluded, for $\nabla_1 U$ is a product of even order while $\nabla_1{}^2 U$ is of odd order.

In a condensed and powerful note on the motion of a rigid system left to itself without external force in a flat hyperspace,* the idea of conservation of distance, which is made fundamental above, is not introduced. He develops kinematics on the basis of his polar unitary system as follows. A point is represented by the polar vector $\rho = \Sigma \iota_h x_h$, and its velocity enters through equations of type $\dot{x}_h = \Sigma p_{hk} x_k$, where $p_{hk} = 0$, $p_{hk} = -p_{kh}$; if, then, $-2p = \Sigma \iota_h \iota_k p_{hk}$, the velocity of the point is $\dot{\rho}$, equal to the vector part of $p\rho$. He employs these formulæ to obtain the Eulerian equations of motion referred to axes travelling with the system, and therefore (be it remarked) self-contained without relation to external bodies; and he points out how their integrals are to be expressed in terms of the time by adaptation of the periodic relations subsisting in the groups of general Θ functions. Thus the isolated rigid system in free rotation, relative to itself alone, provides its own absolute scale of time. The motion is evolved from instant to instant by means of the angular velocities of the body, referred to its own principal axes; just as a spacial extension is continued on Riemann's ideas from element to element by means of the values of its curvatures, or as a field of physical activity is continued stage by stage in space and time in the recent theories of relativity. If, as above, the rotational velocity is represented by time gradients of the changing polar units, referred to their own instantaneous positions, the p_{hk}, ... can be expressed as time gradients of l, m, ...

Clifford published in 'Nature' in 1873 a translation of Riemann's discourse on the construction, or rather continuation, of a space in terms of knowledge of its curvatures at the boundaries of its expansion; and, as Prof. Smith remarks, he "had imbibed the views set forth in it as a part of his intellectual nature." He had already contributed to the Cambridge Philosophical Society in 1870 a note of enthusiastic anticipation,† "On the Space-Theory of Matter," in which the atom was to be merely a mobile deformation of space just after the manner of the recent relativity formulation of gravitation.

* 'Math. Papers,' XXV (1876), pp. 236–240.

† 'Math. Papers,' V. Riemann had already speculated that space need not be uniform around the molecules of matter.

On Gravitational Relativity.

[*Added November* 20.—The symbolic geometrical calculus which is the foundation of the previous discussion pertains only to the homaloid or flat continuum, in which the element of length $\delta\sigma$ is expressed by the formula

$$\delta\sigma^2 = \delta x^2 + \delta y^2 + \delta z^2 + \delta w^2, \qquad w = \iota ct.$$

The phenomena of gravitation have been included by Einstein in this Minkowski scheme by altering slightly the expression for $\delta\sigma^2$, but so that it becomes expressed by the general quadratic function of the elements of length. This generalisation can still be brought within the range of the Clifford geometry by introducing into the analysis a new dimension (ξ) preferably of space; so that

$$\delta\sigma^2 = \delta x^2 + \delta y^2 + \delta z^2 + \delta\xi^2 + \delta w^2, \qquad w = \iota ct,$$

with, of course, an additional component in the relevant vector potential U. For the relations of the electrodynamic field, those above expressed as existing between the components of the vectors $\nabla_1 U$ and $\nabla_1^2 U$, are expressible, whatever be the number of dimensions, as relations of circulation* of the type of Stokes' theorem, after the manner of Ampère's and Faraday's electrodynamic relations, which equate a flux across any sheet (binary locus) in the continuum to a circulation around its (linear) edge. Such relations, if true for the continuum as a whole, are valid also for any continuum of lower dimensions that is included within it. Now any

* *Cf.* an analytical memoir by R. Hargreaves, 'Trans. Cambridge Phil. Soc.,' vol. 21, p. 107 (1908), which involved the germ of the general relativity relations as expressed in this manner. It seems, however, to be sufficient proof to observe that, on account of the vectorial character of the present analysis, on passing to a continuum of lower dimensions, every relation that involves only components of vectors, and their gradients, which exist wholly in that lower continuum, must remain true. [On working out the vectors in the electrodynamic fivefold (with interesting result), this statement proves to be true only for a flat section of it, as closer attention to the analogy of three and two dimensions would have warned. Thus the electrodynamic scheme, as well as gravitation, is modified from the usual form, but actually to a very slight degree. This analytic representation of the world-process as located in a curved section of a flat fivefold, involves, as now appears, other variables which are latent after the manner of the ignored co-ordinates of Routh and Kelvin in dynamics; and it is a question how far it agrees with Einstein's scheme. His method, *cf.* "Die Grundlage . . .," 'Ann. der Physik,' vol. 49, p. 812 (1916), is to transform the known electrodynamics of the flat fourfold into a curved fourfold determined so as to absorb gravitation, as if it were merely another phenomenon added on. But this electrodynamic construct may by Minkowski relativity be situated anyhow in the fourfold; hence its curved transformation ought, by the same relativity, to be situated anyhow in a flat fivefold which is a continuation of that fourfold. If this be allowed, there ought to be no disagreement, though this ultimate electrodynamic invariance is possibly hardly in strictness assured by Einstein's process.]

continuum of four dimensions, having a quadratic line-element, however complex, is expressible as a hypersurface in this homaloid continuum of five dimensions.* If these considerations are correct, the Einstein generalisation, made with a view to include gravitation within his four dimensions, must be interpretable as the geometry of some type of hypersurface constructed in this extended homaloid of five dimensions. For the previous homaloid theory of Minkowski which ignored gravitation, this hypersurface, existing in the five dimensions, in which the world-process is represented, is flat; or more conveniently in some connections it may be taken as a closed region (hypersphere) of assigned uniform extremely small curvature, instead of the unlimited hyperplane. The problem then is to include in the scheme the influence—actually very slight in realizable cases—of gravitation; and this is to be done by recognising slight local deformations on this hypersphere in order to represent that effect. Now in the four-dimensional Minkowski map of the historical world-process, the rays of radiation are the curves of minimum length on the locus for which the analytic element of length $\delta\sigma$ vanishes; and the paths of particles when gravitation was neglected were the curves (then straight lines in the flat) for which the length between assigned terminal points is minimum. If the hypersurface, which is very nearly uniform of very small curvature in the actual problem as presented in nature, can be so chosen that these two relations persist—namely, that the rays of light shall be geodesics on the locus determined by $\delta\sigma$ vanishing, and the free orbits of particles with gravitation now introduced shall be the paths of minimum length on the hypersurface—then one way of absorbing the universal phenomena of gravitation, into the mixed space-time scheme which has arisen from and has transcended and obliterated the previous idea of relativity of positions and motions, will have been accomplished.

Viewed from this angle, the problem of the inclusion of gravitation, along the special lines of Hamiltonian variational dynamics on which a solution has been sought by Einstein, is one of map-making in these hyperspaces. We may illustrate by the simplest example, the one which originated the analytical theory of mapping. A map of the earth's spherical surface constructed on a plane expresses correspondence, point for point; but in other respects the correspondence is necessarily incomplete; for example, if angles are made to agree as in the familiar Mercator projection, the scale of correspondence of lengths must vary from place to place. The representation by a flat map cannot be complete; it can be achieved only as regards

* This may be compared with Clifford's own development of his calculus, which subsumes elliptic or hyperbolic geometry of three dimensions under Euclidean geometry of four dimensions.

correspondence of certain chosen relations, and that only within a very limited range.

Now the problem developed by Einstein may be visualized as in a way the converse of this one. In the Minkowski four-fold continuum of space and time, which may be conceived as a flat or homaloid locus in our present auxiliary five-fold continuum, the scheme of electrodynamics and radiation exists, but the ordinary scheme of gravitation proves not to be adapted to it, when matters are tested by observation of nature to extreme refinement. Here existence means that its essential properties are independent of any particular scheme of measurement of them; as for instance the existence of a solid body in ordinary space is independent of the frame of reference with regard to which a measured survey of its form would have to be conducted: the source of this idea, which is the antithesis of relativity, is presumably our experience of the free mobility, without change, of the solid bodies around us. As just stated, in the flat or homaloid four-dimensional continuum of Minkowski, the phenomena of gravitation do not satisfy this criterion; the problem is to replace it if possible by some less simple type of four-dimensional continuum, constructed most conveniently as a hypersurface within our auxiliary flat five-dimensional scheme, in which both the electrodynamic and the gravitational theory shall exist in the sense above expressed.

The Minkowski world-process is mapped on a flat hypersurface, which can be conceived as existing in this hyperspace of five dimensions, but the gravitational orbits, though definite, are outside it, in the sense that they do not remain invariant, that their mode of specification has to be altered, when the axes of measurement in the hypersurface are changed. Can, then, this flat hypersurface be replaced by another one, so far as regards the actual problem of nature still very nearly flat, merely with very slight deformation in the neighbourhood of (so-called) gravitating masses, so that, while the invariance of electrodynamics is preserved, the same property shall be acquired by gravitation? One mode of acquiring it is that the free path of a particle, which, when gravitation was ignored, was always a shortest line on the flat, shall now be always a shortest line on the new hypersurface when gravitation is included. This is the correlative of the following problem in ordinary map-making: a representation of a given spacial configuration is constructed on the flat which is true to the configuration as regards an assigned limited number of properties; is it possible, by constructing the representation on a surface not flat but nearly flat, to make it true to the original as regards one more property?

The problem of Einstein in its widest generality, as one gathers, is not yet solved. But in the case that alone is amenable to practical test, in

which the adjustments are extremely minute, an approximate verification ought, of course, to be possible; and this is what Einstein has carried through, threading his way with great mastery of conception and analysis. The result obtained is that one can in this manner bring gravitation into the four-dimensional scheme, which is independent of axes of co-ordinates, at the cost of doing some violence to electrodynamics by introducing complications, to a degree, however, that is far too minute for experimental or observational scrutiny, and by doing violence to gravitation itself and its relations to light that happen to be just within the range of detection in three cases. It is as if one found that a cap would not fit exactly over a surface, but that a close fit could be forced by slight local stretching.

It is open to a critic to urge that this way of getting gravitation into the electrodynamic scheme proves nothing by itself, as Einstein apparently recognises, as to the unrestricted necessity or validity of a mixed space-time representation of the world-process; for it merely verifies some exceedingly small adjustments only to the first order of approximation, while a cosmological principle which aspires to the dignity of a law of thought must be absolutely true, without regard to approximation, for all magnitudes of change. That is, it would have to remain true if astronomical verification could deal with relative motions of many thousands of miles per second, instead of as actually with only a few tens. Viewed as demonstration, it is merely an approximate adjustment, and so carries very little evidence of universal unrestricted validity. Its true test had to wait some years for the opportunity of observation of nature. The adjustment of the slightly anomalous motion of the perihelion of Mercury taken by itself might well have been merely a coincidence. But, if in addition a prediction of the exact amount of deflection of a ray of light by the solar gravitation has been verified by the observers of the recent eclipse, as appears to be the case, then the point of view has most probably been established, though not necessarily in its present form. Even if the other prediction, of observable increase in the wave-lengths of light emitted in a field of gravitation, such as the sun or a star, proved to fail, the point of view must still claim attention: and the problem would be to amend the formulation once more, in one of the ways which may still be open, so as to force the scheme to include one other feature.

These considerations are submitted tentatively, with a view to getting far enough away from the complex analytical details of the gravitation theory to enable a judgment to be formed on its general physical aspect, and its relation to other recognised general principles of the scientific interpretation of nature.

One other remark, already hinted at, may be advanced. These theories arose out of the idea of the relativity of the positions and motions of material bodies. On the theory of an æther that idea was established demonstratively long ago, up to the order to which observation can test it, expressed either in terms of the original natural interpretation that motion through the æther affects slightly the dimensions of material bodies,* or on the new conception of a mixed space-time foundation for representation of the system of nature. But in either case it is a very limited relativity; for it holds as regards translatory motion of the observer's system only when that motion is uniform, and as regards rotational motion not at all. It would appear that exposition of the new theory ought to get rid, as it can, of this glaring imperfection by ceasing to designate it as a theory of relativity at all; one would describe it as the theory of interdependence of space and time, such that time is virtually a fourth dimension interrelated with the other three dimensions of space, and *sui generis* only in so far as its measure is a number that is algebraically a pure imaginary. This statement seems to express the scope of the theory; by the theory both position and motion are transcended, in any sense that practically belongs to these concepts, and there seems to be no relevance in any further discussion about their relativity. The propagation of radiation is transcended also; its velocity becomes merely the dimensional multiplier that is required to make time homogeneous with length.

And again, a common interpretation of its point of view is that there can be no natural frame of reference, no æther. This appears to be almost repugnant to common-sense: for it would make the universe consist at best of a heap of unrelated particles in the void. And, in fact, these abstruse arguments on relativity cannot advance one step without the most elaborate frames of reference. The aim with which theory has to be content is to prove that we can get on equally well with a great variety of modes of expression of the natural frame of reference. It can even be maintained that this result is a strengthening rather than a destruction of the notion of an æther of space; for its aid cannot be dispensed with, while it is proved that its mode of intervention, possibly hitherto imperfectly appreciated, is so fundamental that it can be expressed in terms of a great variety of simple statements without mutual contradiction.

Can a Field of Gravitation Disturb the Free Periods of a Radiation Spectrum?

If this hyperspacial version of the Einstein gravitational theory is a valid presentation of physical reality, it hardly seems to warrant the usual

* This mode of explanation would hardly be open for the recent astronomical verifications.

further conclusion that the free periods, as observed terrestrially, of the radiations emitted from the sun or a star, should be displaced towards the red end of the spectrum. At any rate further elucidation seems desirable: and with this in view we may strengthen our intuitional bearings by closer scrutiny of a parallel correlation applicable to spaces of two and three dimensions.

Geometry on a given curved surface is self-determined, in the manner developed by Gauss, in a binary set of curvilinear co-ordinates (p, q) and the expression for the element of length δs in the form

$$\delta s^2 = f\delta p^2 + 2g\delta p\delta q + h\delta q^2$$

in which f, g, h are known functions of position on the surface, and so are functions of (p, q). If we can effect a transformation

$$\xi = \mathrm{F}_1(p, q), \qquad \eta = \mathrm{F}_2(p, q),$$

so that the expression for the element of length becomes

$$\delta s^2 = k(\delta\xi^2 + \delta\eta^2),$$

where k is of course a function of ξ, η, then the equation for the shortest paths on the surface, which is $\delta\int ds = 0$, becomes

$$\delta\int k^{\frac{1}{2}} d\sigma = 0, \text{ where } \delta\sigma^2 = \delta\xi^2 + \delta\eta^2.$$

The quantities $(\xi\eta)$ may thus be taken as the co-ordinates of a correlative point in a plane: and a correspondence is established between the points (pq) on the curved surface and the points $(\xi\eta)$ on a flat sheet which is in certain respects a map of it, corresponding elements of area being similar but not equal. A geodesic or shortest line on the surface corresponds to the orbit of a particle m on the plane, in a field of force whose potential W is determined by the equation of conservation of energy

$$\tfrac{1}{2}mk + m\mathrm{W} = \mathrm{E}_0;$$

for the variational equation has been transformed* into the equation of Least Action $\delta\int v d\sigma = 0$ in the field of force for which $v = k^{\frac{1}{2}}$. Correlative elements of length near any point are in the ratio $k^{\frac{1}{2}}$, and the elements of extension are similar in this ratio.

Now imagine, after the geometers, a two-dimensional intellect whose activities are confined to this curved surface. He may form a scheme of his surroundings either in terms of the co-ordinates (p, q) measuring the actual

* As regards such correlations, see two papers, "On the Immediate Application of the Principle of Least Action to Dynamics of a Particle, Catenaries [Rigid Dynamics, Hydrodynamics], and other related Problems," 'Proc. London Math. Soc.,' vol. 15, (1884), in which this principle is worked out with examples. *Cf.* also the general analytical theory in Darboux, 'Théorie générale des Surfaces,' vol. 2, chap. vi (1889).

curved space, or in terms of the other co-ordinates (ξ, η) specifying more directly the correlated flat space. When the curved space is thus represented, the shortest lines on it are transformed into dynamical orbits on the flat. One might propound a problem, what should be the form of the surface in order that the orbits in the plane should belong to a field of gravitation? This seems to be an analogue, in two spacial dimensions, of the procedure of Einstein, except that possibly k has to be unity also. For it is to be remarked that what the intellect thus limited has to deal with is confined to the general type of connections, the mode of continuity, in the extension in two dimensions: there is no reason except convenience for a choice as to whether he should refer the binary manifold to a geometric frame of reference (p, q), or to another (ξ, η); the latter is here the simpler frame, while the representation of the system is simpler on the former one. But when the intellect, originally limited, has learned to expatiate into the flat space of three dimensions, which we are familiar with as our own natural representation of space, things will have gained a wider outlook, and relations between the two modes of representation in terms of (p, q) and of (ξ, η) will have opened out.

Now let us transfer these considerations to the more complex physical problem. We have to contemplate an intelligence to whom the Minkowski continuum of mixed space and time of four dimensions, one of them imaginary, is intuitive. To him the historical world-process is spread out as one configuration situated in this Euclidean continuum. Whereabouts it is situated in it and how it is orientated is indifferent, just as the position of a material system of assigned internal constitution in ordinary space is indifferent; and that is what relativity has been reduced to in this scheme. It is at the choice of our four-dimensional intelligence to refer the historical world-process thus expanded before him, either to a flat fourfold space of reference analogous to (ξ, η) in the illustration, the space of Minkowski, having then to recognise gravitational forces and orbits modified under their compulsion within it, or else with Einstein to try whether it can be referred to a curved heterogeneous fourfold space of reference typified by (p, q) in the illustration, so that gravitational orbits shall become simply straightest lines so far as may be, and the physical notion of acceleration ascribed to a universal type of force thus replaced by a foundation purely geometrical or rather kinematic.

Neither of these representations is more valid than the other. To see them in direct relation to each other they must be surveyed by an intelligence to whom a five-dimensional mixed space-time scheme is intuitive, in which they are both contained as the curved surface and the plane of the illustration are

contained in space of three dimensions. In the Einstein analysis direct intuitive correlation is replaced by the surer method of algebraic correspondence. The co-ordinate system $(xyzt)$ for the flat hyperspace corresponds say to $(x'y'z't')$ for the curved hyperspace, just as (ξ, η) corresponded to (p, q), except that k need not be explicit. Their relation as determined by Einstein to the first degree of approximation that is sufficient for ordinary gravitation appears to be that, so far as regard the orbits, the constitutive spacial equation for the usual convention of flat space

$$\delta\sigma^2 = \delta x^2 + \delta y^2 + \delta z^2 - c^2\,\delta t^2$$

corresponds to another for the presumed actual curved space,

$$\delta\sigma^2 = \delta x'^2 + \delta y'^2 + \delta z'^2 - c'^2\,\delta t'^2,$$

where $\quad c'^2 = c^2 g_{44}, \qquad g_{44} = 1 - 2c^{-2}\mathrm{V}$ where $\nabla^2\mathrm{V} = -4\pi\rho$.

A geometrical quantity V arises of the type of a gravitational potential determining the density ρ, which latter may or may not prove to belong to a completely invariant mass: and $\frac{1}{2}c'^2 = \frac{1}{2}c^2 - \mathrm{V}$ as if the light also had a mass subject to gravitation.

The curved fourfold continuum is of the Riemann type, being flat as regards its infinitesimal elements of extension, as an element of an ordinary curved surface is flat.

We can compare events occurring in the element $\delta x'\delta y'\delta z'\delta t'$ of the supposed actual curved continuum of Einstein as thus situated in the flat five-dimensional region, with events in the image of it which we form, in the corresponding element $\delta x'\delta y'\delta z'\delta t'$ in the flat four-dimensional region in which a strained representation, involving gravitation somewhat modified,* has been made. For these infinitesimal elements are both practically flat and so are comparable. The scale of ordinary space is not disturbed in the representation, but the scale of time is changed. For in the fivefold natural and presumably universal because simpler frame of reference, a Euclidean space-time continuum, c has its standard value: in the latter representation strained to a flat continuum of lower order, and so involving heterogeneous features interpreted as gravitation, it is changed to c'. But events are identical in the two representations, both flat and so directly comparable as regards these infinitesimal elements; hence as $\delta x'$ is equal to δx, so $c'\delta t'$ must be equal to $c\delta t$, where, as above, $c'/c = 1 - \mathrm{V}/c^2$. The scale of the apparent time t', involved in the conventional flat representation that is interpreted as a gravitational system is therefore different from that of

* The modification of gravitation, and half its influence on rays of light, arise in the second approximation, in which, in a symmetrical field, the radial element $\delta r'^2$ in the value of $\delta\sigma^2$ becomes affected by a factor $1 + 2c^{-2}\mathrm{V}$. [This arises from an added postulate of conservation of fourfold extent: see a paper in 'Monthly Notices R. Astron. Soc.']

the standard time t, applicable to the world-system treated as non-gravitational and located as a four-dimensional non-flat configuration in the flat five-fold space. Where V is greater, c', the apparent velocity of propagation in the flat element, is smaller, and corresponding apparent times are greater in inverse proportion. Thus the apparent periods of vibration of a hydrogen molecule in the field of high gravitational potential at the sun are greater than their normal periods; but on account of the smaller apparent velocity of propagation c', the wave-lengths of the radiation emitted from it remain normal and constant throughout its course. When this radiation has travelled to a place of null gravitational potential, its apparent velocity will have fallen to the normal value c, and its period also will thus have become normal. Thus, if the superior intelligence that can visualize the five-dimensional manifold in which the true transcendental system and its gravitational representation both subsist, makes a comparison between them, he will recognise that they are merely referred to different scales of time, so that events appear to change more slowly in a region in which there is a higher gravitational potential when flatness is asserted and the fiction of gravitation therefore introduced. Thus, possessing his standard of normal time of the fivefold space, he will recognise that a molecule of hydrogen vibrating in a region near the sun emits radiations of slightly longer periods, as apparent in the gravitational scheme, than a like molecule vibrating in a terrestrial laboratory. This super-intelligence, with his transcendent powers of comparison, will thus recognise on the forced flat interpretation of the universe, a gravitational influence on the periods of free vibrations of a molecule of given material, which is eliminated when the real kinematic curved representation replaces the apparent flat and gravitational one. But the opportunities open to him are not those available to actual mundane spectroscopic verification: what is there immediately determined is not the period of vibration of a hydrogen molecule in the sun—that is beyond our reach—but the period of the waves emitted by it *as these waves pass the earth.* As the gravitational potential near the earth is comparatively small, and does not change much in the course of its annual motion, it would seem, at any rate subject to correction, both that observed spectral periods may be taken as constant throughout the year, and that as observed in the same locality there is no difference between light from the sun and light from a terrestrial source. Because in corresponding small elements of the related fourfold extensions spacial measurements agree, and the events are identical, all times of the systems must be correlated also, but in a ratio which varies from element to element.

The object of merging gravitation in a strain of the mixed space-time frame would thus be reduced to securing relativity, in the highly sublimated sense

that the four-dimensional non-flat construction which presents the world-process at a glance is independent of its position or orientation in the five-dimensional flat electrodynamic construct, or æther as we could name it, in which it is contained, as a surface is contained in ordinary space. But on forcing this presumably more ultimate formulation into a four-dimensional scheme of the simple flat type, there proves to be a residual want of fit which cannot be got rid of, and has to be recognized as a new feature or property of nature, namely, gravitation in a very slightly modified form, which thus interferes slightly with the rays of light and even with the fundamental electrodynamic relations. But it does not seem to be clear as yet, unless the present expository process turns out to be a failure, that the periods of natural radiation can properly be asserted to be modified by a gravitational field in any observable manner. As remarked for the introductory illustration, the linear element δs was there not necessarily invariant. In the present type of exact hyperspacial correspondence, it is just such invariance that causes the strained representation involving a changing scale of local time. In the application to actual gravitation the factor analogous to k is taken at its very approximate value unity. If it could be exactly unity (actually the relation is even not quite isotropic) this invariance of the hyperspacial element of length would provide perhaps a portable infinitesimal measuring rod for comparisons, which would not alter its length when its direction is changed, thus making comparisons thinkable without requiring the merging of the system in the wider auxiliary Euclidean spacial system of five dimensions as has here been done.

Wider possibilities of syntheses with k different from unity arise, but they would be more complex, vitiating the directness of correlation with the flat time-space scheme that is assumed, perhaps only provisionally, in ordinary physics, and possibly doing further violence to the latter and to the electrodynamic relations which it involves.

Thus we postulate a fivefold electrodynamic potential and its concomitant electrodynamic vector-systems in the Euclidean auxiliary space $(x, y, z, \xi, \iota ct)$. Then any section of this space and its vector-system is a hypersurface of four dimensions of the same Minkowski type as that system itself, and represents a possible electrodynamic world-process; including implicitly its gravitation, which would become apparent only when the hypersurface, actually already nearly flat, is forced into representation on a hyperplane.

It is, of course, a very striking feature that Einstein's theory of gravitation not merely forces the Newtonian law into the impress of a relativity mould, but that it even evolves that very law in the form of its Laplace-Poisson characteristic equation, from a relativity representation in fivefold Euclidean

space which does not contain any such extraneous feature. But also, viewed from the other side, in extensional analysis of this type appropriate to isotropic Euclidean manifolds, the vector operator ∇ and its powers are the fundamental ones, so that it is not really very surprising that gravitational analysis can be linked up with a theory of deformable space.

Finally and again, this re-statement of theories of relativity as relations of correspondence in space and time, by aid of uniform auxiliary manifolds of higher dimensions, may appear retrograde: in the earliest phase relativity was just such correspondence.* But it has the advantage of getting rid of the very puzzling auxiliary apparatus of local timekeepers, and their changes of rate when moved about. And, moreover, it is not, in fact, possible to do without a scheme of space and time; relativity merely asserts in various ways that its final specification so far eludes our powers that a large number of partial modes of specification can be employed indifferently over a wide range of problems.

On the Variation with Frequency of the Conductivity and Dielectric Constant of Dielectrics for High-Frequency Oscillations.

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1. *Introduction.*

Although we have a certain amount of knowledge regarding the variation of the conductivity of dielectrics with frequency for comparatively low frequencies, within the telephonic range, say, up to 5000 per second, where the conductivity is in general a linear function of the frequency, it cannot be said that any information exists at present as to what happens when we extend the range of frequencies up to those employed in radiotelegraphic work. That energy is dissipated in condensers used in oscillation circuits has been known since 1861, when W. Siemens† pointed out that the glass of a Leyden jar became heated on charge and discharge. Threlfall,‡ extending

* *Cf.* 'Æther and Matter,' chap. xi. (1900). Here in cognate manner the five-dimensional space-time foundation is introduced in order to provide the necessary standards of time and space, which, even though provisional, are indispensable.

† 'Berlin Akad. Monatsber.,' October, 1861.

‡ 'Phys. Rev.,' vol. 4, p. 57, and vol. 5, pp. 21 and 65.